Self-organization and anomalous diffusion

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The self-organizing process is investigated in the continuum limit of the cellular automaton model introduced by Bak, Tang and Wiesenfeld. An anomalous diffusion equation is proposed for the description of this process, and an analytical method for the solution is presented in one dimension, based on an adiabatic approximation.

Recently, there has been a great interest in various models that display self-organized criticality (SOC). The term "criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena. "Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics. Bak, Tang and Wiesenfeld (BTW) [1] suggested that there may be an intimate connection between scale invariance in the spatial and the temporal domains, i.e. between the fractal shapes in nature [2] and the $1/f^a$ noise [3]. As an illustration, BTW introduced a sandpile model [1] based on a cellular automaton algorithm. Avalanches generated by an external perturbation can be observed on all length and time scales, and the characteristic distributions obey power-law behavior.

An interesting question is the study of the time evolution of an arbitrary initial state towards the SOC, i.e. the self-organizing process. We show that the dynamics of the BTW model is the discretized version of a specific anomalous diffusion equation, whose density dependent diffusion coefficient has a pronounced peak at some critical value. As a result, the system organizes itself into a homogeneous state where the density takes on the critical value, resulting in the supersensitive behavior where all perturbations relax quickly (SOC).

At first, we describe shortly the original BTW sandpile model [1] on an $L \times L$ square lattice, using Dhar's formalism [4]. Let us label the sites by integers from 1 to $L^2$. To each site $i$ is assigned an integer height variable $z_i$. Select a site at random and increase $z_i$ by 1, while leaving the other sites...
unchanged. A toppling event occurs if the height at a site exceeds a prescribed critical value $z_c$ (such a site is called an activated site). On toppling at site $i$,

$$z_j \to z_j - M_{ij} , \quad j = 1, \ldots, L^2 ,$$

(1)

where $M$ is the following $L^2 \times L^2$ matrix (the so called toppling matrix):

$$M_{ij} = \begin{cases} +4 , & \text{if } i = j , \\ -1 , & \text{if } i \text{ and } j \text{ nearest neighbors} , \\ 0 , & \text{otherwise} , \\ \end{cases}$$

(2)

which can be decomposed into an $L \times L$ hypermatrix consisting of the one-dimensional toppling matrix and the identity matrix. Note that (2) is a discretized version of the Laplacian. The properties of $M$ ensure that on toppling at site $i$, $z_i$ must decrease, $z_j$ for $j \neq i$ cannot decrease, and there is a local conservation rule, i.e. neither sources nor sinks are present in the system. The local conservation rule is not a necessary condition for the SOC behavior [5]. "Height" can leave the system through the boundaries, otherwise no steady state is possible. It is assumed that any configuration relaxes to a stable configuration in a finite number of steps.

In order to give a hint for the connection of the cell automaton algorithm with diffusion processes, let us rearrange eq. (1) in the following form:

$$z(t+1) - z(t) = -Me(t) , \quad e_i(t) = \Theta(z_i(t) - z_c) ,$$

(3)

where $z(t)$ denotes the full configuration at time $t$, $\Theta$ is the Heaviside step-function, and $e(t)$ points onto the activated sites. When the length of the discrete time interval between two successive steps goes to zero, the left-hand side of eq. (3) becomes proportional to the first time derivative of the height variable. On the right-hand side $M$ becomes the Laplacian $\Delta$. This suggests that the continuum version of eq. (3) is

$$\frac{\partial \rho(r)}{\partial t} = \Delta[\Theta(\rho(r) - \rho_c)] .$$

(4)

We have introduced here the continuous "height-density" $\rho(r)$ with the critical value $\rho_c$ instead of the discrete height variable $z$.

As a result, we get an anomalous diffusion equation

$$\frac{\partial \rho}{\partial t} = \text{div}[D(\rho) \text{ grad } \rho] ,$$

(5)

where $D(\rho)$ is a density dependent diffusion coefficient. In order to get rid of
the singularities coming from the Heaviside step-function, $D$ has to be regularized by taking e.g. the Lorentzian shape

$$D(\rho) = \frac{D_0}{1 + \left[ (\rho - \rho_c)/\epsilon \right]^2}, \quad (6)$$

where $\epsilon$ is some small parameter characterizing the width of the peak. We stress that the only relevant feature of the diffusion coefficient $D(\rho)$ is that it has a sharp maximum at the critical value.

The physical picture is the following. At a given time, one may distinguish three different spatial regions: a critical region, where $\rho \approx \rho_c$, a supercritical region, where $\rho > \rho_c$, and a subcritical one, where $\rho < \rho_c$. In the sub- and supercritical regions the diffusion coefficient is very small. On the supercritical–critical boundary the diffusion coefficient grows steeply, and there is a large flux which enters the critical region. In the critical region the diffusion coefficient is very large, thus any inhomogeneity disappears “instantaneously”. Consequently, the inflow flux is transported through the critical region to the critical–subcritical boundary, where it enters the subcritical region. The height of the supercritical–critical boundary layer decreases till it reaches the critical value $\rho_c$, when the layer becomes a part of the critical region. As a result, the boundary moves towards the supercritical region, until this later is completely absorbed by the critical one. The behavior of the other boundary is completely similar, the only difference is that the direction of the flux is opposite. The critical region grows steadily in this direction as well.

The time evolution of the system is determined by two distinct time scales: in the critical region there is a quick diffusion process with characteristic time $\tau_1$, and in the off-critical region there is a much slower relaxation with time scale $\tau_2$: $\tau_1 \ll \tau_2$. As we are interested in the spreading of the critical region, we have to consider the system on a medium time scale $\tau$, characterizing the motion of the boundaries:

$$\tau_1 \ll \tau \ll \tau_2. \quad (7)$$

This fact allows us to develop an effective theory describing the time evolution of the moving boundaries by using an adiabatic approximation. For the sake of simplicity, let us consider a one-dimensional system. Without loss of generality we may take $\rho_c = 0$ and $\rho(x = 0, t = 0) = \rho_c$. In the adiabatic approximation, we take the limits $\tau_1 = 0$ and $\tau_2 = \infty$ (cf. eq. (7)). This means that in the critical region the relaxation process is instantaneous, while in the off-critical region there is no diffusion at all. Consequently, inside the critical region the density profile $\rho(x, t)$ is given by a stationary solution of eq. (5),
determined by the (time dependent) boundary conditions. The equation
\[ \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) = 0, \tag{8} \]
which determines the stationary solutions of eq. (5), can be transformed to the form
\[ \frac{\partial^2 \Gamma(\rho)}{\partial x^2} = 0, \quad \text{where} \quad \Gamma(\rho) = \int_0^\rho D(\rho') \, d\rho'. \tag{9} \]
Eq. (9) has the general solution \( \Gamma = \alpha x + \Gamma_0 \), where \( \alpha \) and \( \Gamma_0 \) are integration constants. (For the sake of simplicity we omit the arguments where they are unambiguous.) Defining the inverse function \( \omega \) of \( \Gamma \) via \( \omega(\Gamma(\rho)) = \rho \), we get for the stationary density distribution: \( \rho_s = \omega(\alpha x + \Gamma_0) \). If we introduce the notations \( \xi(x) = \rho(x, t = 0) \), \( x_+(t) \) and \( x_-(t) \) for the position of the supercritical–critical and subcritical-critical boundaries at time \( t \), then the adiabatic approximation allows us to take the following ansatz for the density profile:
\[ \rho(x, t) = \begin{cases} \omega(\alpha(t)x + \Gamma_0(t)), & \text{if } x \in [x_-(t), x_+(t)], \\ \xi(x), & \text{otherwise}. \end{cases} \tag{10} \]
The variation of the parameters \( \alpha \) and \( \Gamma_0 \) takes place on the intermediate time scale \( \tau \). Using the continuity of \( \rho \) at the boundaries \( x_+(t) \) and \( x_-(t) \),
\[ \alpha(t) x_+(t) + \Gamma_0(t) = \Gamma(\xi[x_-(t)]) = \Gamma_+ , \]
the time dependent parameters can be expressed as
\[ \alpha(t) = \frac{\Gamma_+ + \Gamma_-}{x_+ - x_-}, \quad \Gamma_0(t) = \frac{x_+ \Gamma_- - x_- \Gamma_+}{x_+ - x_-}. \tag{11} \]
Eq. (5) may be thought of as a continuity equation expressing the local conservation of \( \rho \), with a current given by \( j = -D(\rho) \nabla \rho \). Using an integrated version of this conservation rule, we may derive first order differential equations governing the time evolution of the boundaries \( x_+(t) \) and \( x_-(t) \) as follows. The current at the point \( x = 0 \) at time \( t \) is
\[ j(0, t) = -D[\rho(0, t)] \left( \frac{\partial \rho(x, t)}{\partial x} \right)_{x=0} = -\alpha(t), \tag{12} \]
where we have used the ansatz (10) and the well known theorem on the derivative of the inverse function (we assume that the point \( x = 0 \) where the
system was critical at $t = 0$, remains in the critical region during the whole process. Thus the total amount of $\rho$ that has flown through $x = 0$ till time $t$ is $\int_0^t \alpha(t') \, dt'$. But this is equal to the total difference of the initial distribution $\zeta(x)$ and the density profile $\rho(x, t)$ at time $t$:

$$\int_0^t \alpha(t') \, dt' = \int_0^t [\zeta(x) - \omega(\alpha(t) x + I_0(t))] \, dx,$$

where we have used once more the ansatz (10). This is the equation that determines the time evolution of $x_\pm(t)$.

In order to write eq. (13) in a more manageable form, let us introduce the following notations:

$$\Xi_\pm = \int_0^{x_\pm(t)} \zeta(x') \, dx', \quad \Lambda_\pm = \int_{f_0(t)}^{r_\pm} \omega(y) \, dy.$$  \hspace{1cm} (14)

Using these notations, eq. (13) can be simplified to

$$\int_0^t \alpha(t') \, dt' = \Xi_\pm - \frac{1}{\alpha} \Lambda_\pm.$$ \hspace{1cm} (15)

Thus we get an alternative expression for the current density (c.f. eq. (11)):

$$\alpha(t) = \frac{\Lambda_+ - \Lambda_-}{\Xi_+ - \Xi_-}.$$ \hspace{1cm} (16)

From eqs. (11) and (16) we get a functional relationship between $x_+$ and $x_-$. The trivial solution $x_+ = x_-$ should be discarded as unphysical. The knowledge of the functional dependence of $x_-$ on $x_+$ allows us to consider the quantities $\alpha$, $\Xi_\pm$ and $\Lambda_\pm$ as functions of the single variable $x_+$. By taking the time derivative of eq. (15) we get a separable first order ordinary differential equation for $x_+(t)$, whose solution is

$$t = \int \frac{1}{\alpha(x_+)} \frac{d}{dx_+} \left(\Xi_+ - \frac{1}{\alpha(x_+)} \Lambda_+\right) \, dx_+.$$ \hspace{1cm} (17)

A simple example is the case when the diffusion coefficient is given by eq. (6) and the initial density profile is linear. The symmetry of the problem ensures that $x_-(t) = -x_+(t)$ and $I_0(t) = 0$ at all times. From eq. (17), the asymptotic behavior of the boundary is given by the power law
Both the numerical solution [6] of eq. (5) and direct simulations performed on the original BTW model gave results in complete agreement with the above effective theory [7].

In the case when the diffusion coefficient profile has an asymmetric form, an interesting new phenomenon arises. The linear size \( l \) of the growing critical region still follows the power law

\[
l = |x_+(t) - x_-(t)| \sim t^{1/3} \quad \text{for } t \gg 1,
\]

which is the same as in the case of a symmetric diffusion coefficient profile (cf. eq. (18)). But in the asymmetric case \( \Gamma_0(t) \) is different from zero, consequently the critical region’s midpoint \( x_0 \), where \( \rho(x_0, t) = \rho_c \), creeps towards the region of lower diffusion coefficient. Surprisingly, \( x_0(t) \) also follows a power-law

\[
x_0(t) \sim t^{2/3}.
\]
We plotted in fig. 1 the time evolution of the density profile in one dimension, for a linear initial density distribution. The diffusion coefficient was

$$D(\rho) = D_0 \left[ \arctan \left( \frac{\rho - \rho_c}{\epsilon} \right) + \frac{\pi}{2} + \frac{\epsilon \rho_c}{\epsilon^2 + (\rho - \rho_c)^2} \right],$$

(21)

whose shape is shown in fig. 2 (the narrow asymmetric peak belonging to $t = 0$). Fig. 2 shows the time evolution of the diffusion coefficient's profile. The insets show the above mentioned power-law behaviors.

Although it is known that the BTW model in one dimension does not exhibit critical behavior [8], we have found that the self-organizing process itself obeys power-laws. This feature is a direct consequence of the shape of the density dependent diffusion coefficient, which has a narrow, pronounced peak at some critical value.

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References