Introduction to Financial Mathematics

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2011, ELTE

Outline

- Financial mathematics in general, and in market modelling
- Introduction to classical theory
- Hedging efficiency in incomplete markets
- Conclusions

Mathematical apparatus

- Probability theory
- Stochastic processes
- Statistics
- Ordinary differential equations
- Partial differential equations
- Linear algebra

Mathematical apparatus

• Probability theory	OK
• Stochastic processes	OK ?
• Statistics	OK
• Ordinary differential equations	GOOD
• Partial differential equations	GOOD
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• Numerical algorithms	
• Modeling	

Purpose

- Historical analysis, prediction
- Simulation, risk-analysis

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- Build models that enables consistent pricing of ‘simple’ and ‘complicated’ financial products

A gambling problem

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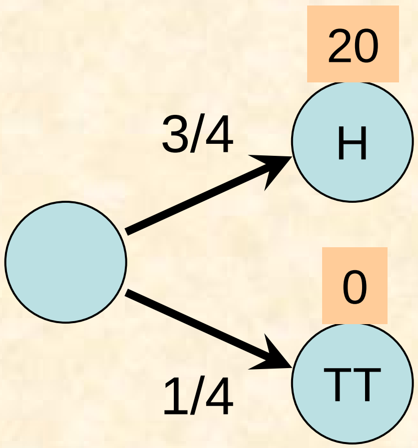

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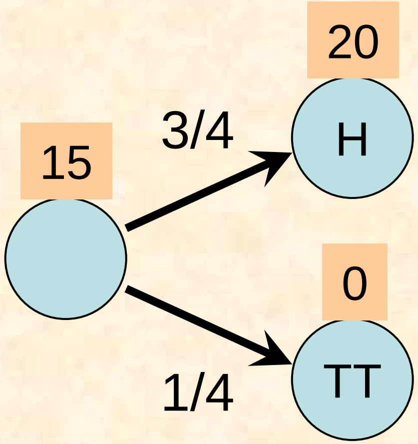
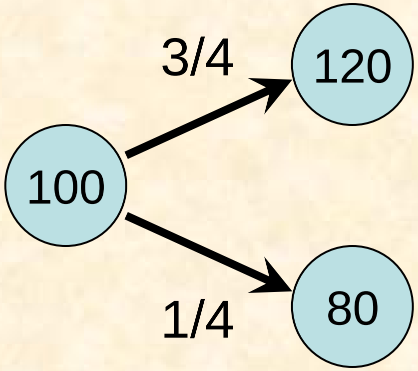


The correct price for this game is 15$

Another gambling problem

The price of a stock is 100$. You can buy a call option on this stock with strike 100$. This means that you have the option to buy the stock for 100$ tomorrow. The stock price tomorrow is 120$ with probability ¾, and 80$ with probability ¼ .

How much would you pay to have this option?



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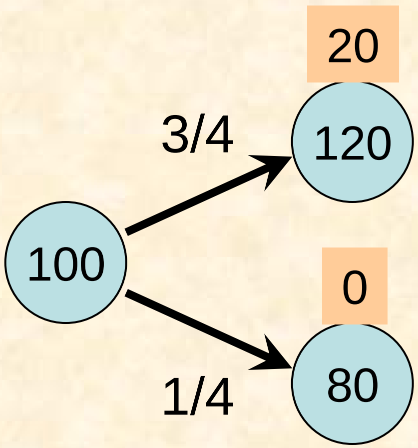

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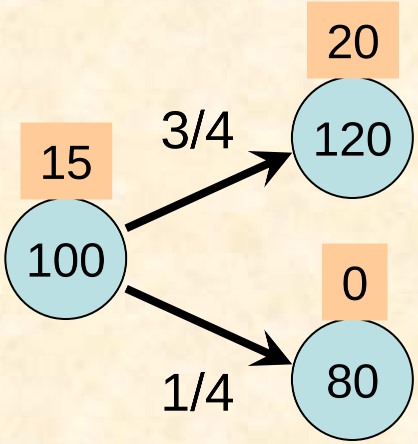

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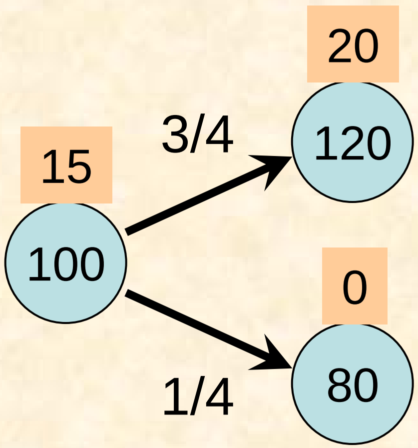

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WRONG !!!

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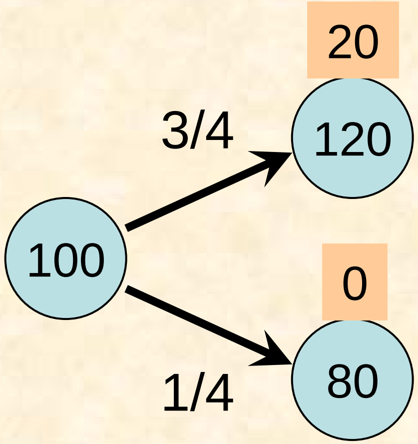

Sell ½ stock for 50$.

If it goes up to 120$, collect 20$ from option and buy back ½ stock for 60$.

In this case, you win 10$.

If it goes down to 80$, option is worthless, and buy back ½ stock for 40$.

In this case, you win 10$.

With zero risk, you made 10$!!!

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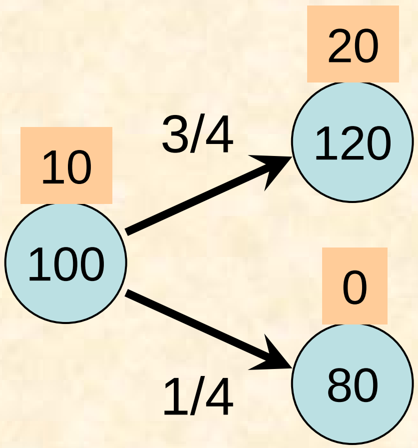

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So the correct price must be 10$

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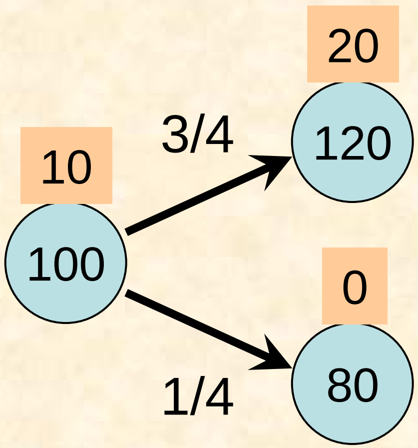

Any different price would mean arbitrage.

If someone quotes a lower price, we buy option, hedge, and realize profit with zero risk.
If someone quotes a higher price, we sell option, do the inverse hedge, and again realize profit with zero risk.

So the correct price must be 10$

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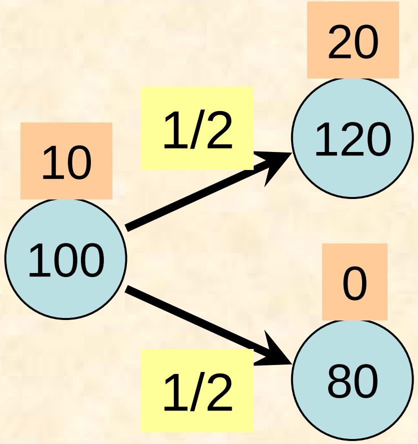


Any different price would mean arbitrage.

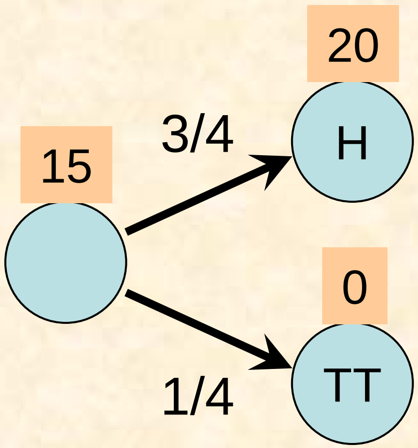
We need a new measure to calculate the correct price of the option. This measure does not depend on our assessment of chances, it only depends on the prices of the underlying in the different scenarios.

So the correct price must be 10\$

Comparing the two gambling problems

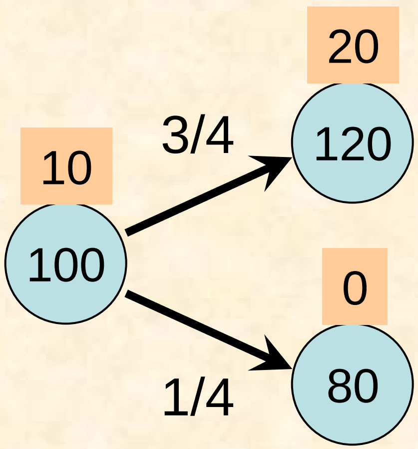
Coin flip

The correct price for this game is 15$



Call option

So the correct price must be 10$



Why are the prices different ?

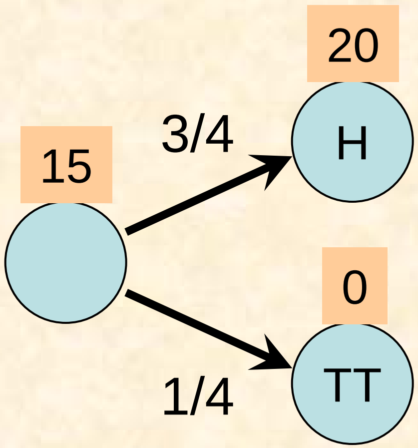
Comparing the two gambling problems

Coin flip

Call option

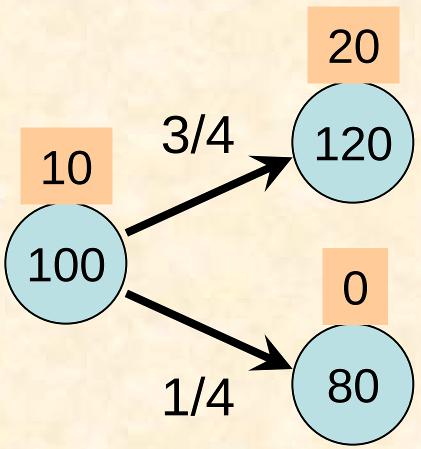
The correct price for this game is 15$

So the correct price must be 10$

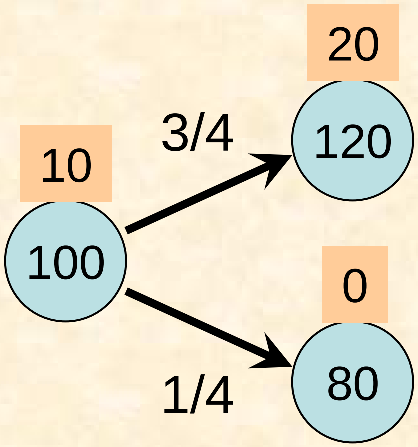


Positive expected value, nonzero risk

Lower expected value, zero risk



Why buy an option?



Speculative
investment strategies

	1 stock / $100		
Spend	$100		
Profit (3/4)	$ +20		
Loss(1/4)	$ -20		
P&L	$ +10		
P&L %	10%		

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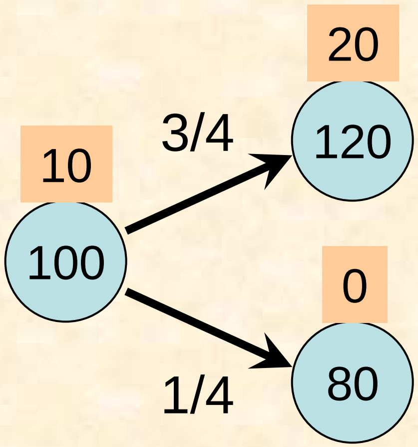
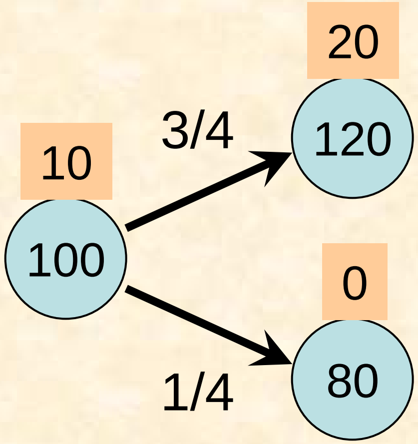


Speculative
investment strategies

	1 stock / $100	5 stocks / $100	
Spend	**$100**	**$100 + $400**	
Profit (3/4)	**$ +20**	**$ +100**	
Loss(1/4)	**$ -20**	**$ -100**	
P&L	**$ +10**	**$ +50**	
P&L %	**10%**	**50%**	

Why buy an option?

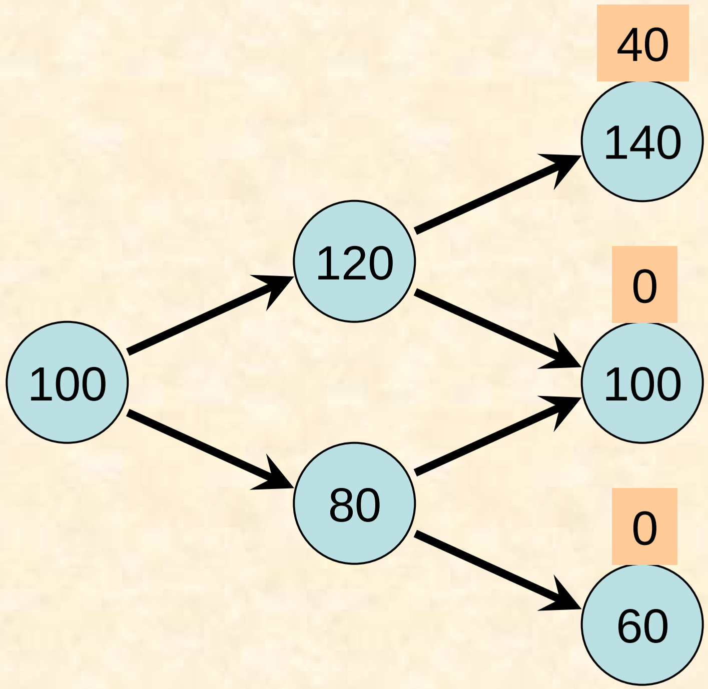


Speculative
investment strategies

	1 stock / $100	5 stocks / $100	10 calls / $10
Spend	$100	$100 + $400	$100
Profit (3/4)	$ +20	$ +100	$ +100
Loss(1/4)	$ -20	$ -100	$ -100
P&L	$ +10	$ +50	$ +50
P&L %	10%	50%	50%

Binomial Tree Model

Call option with maturity at the second tic, with strike of 100$



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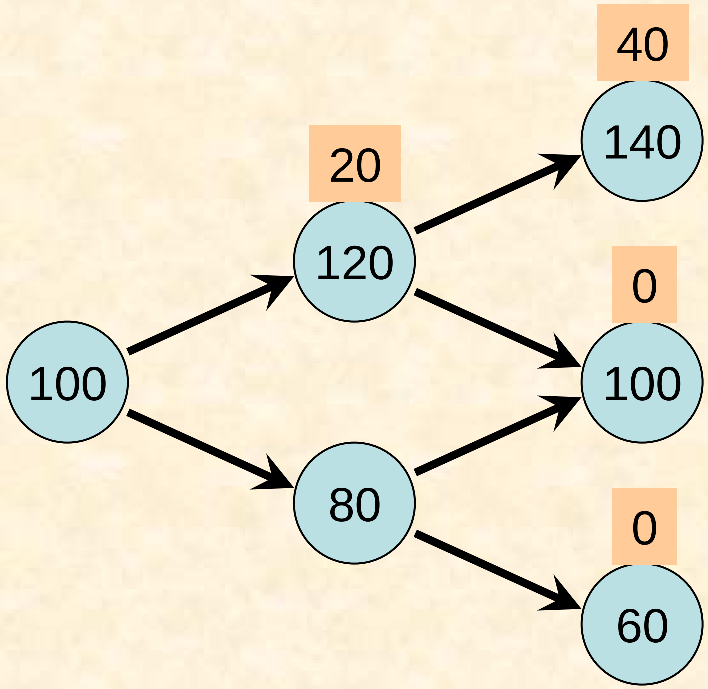
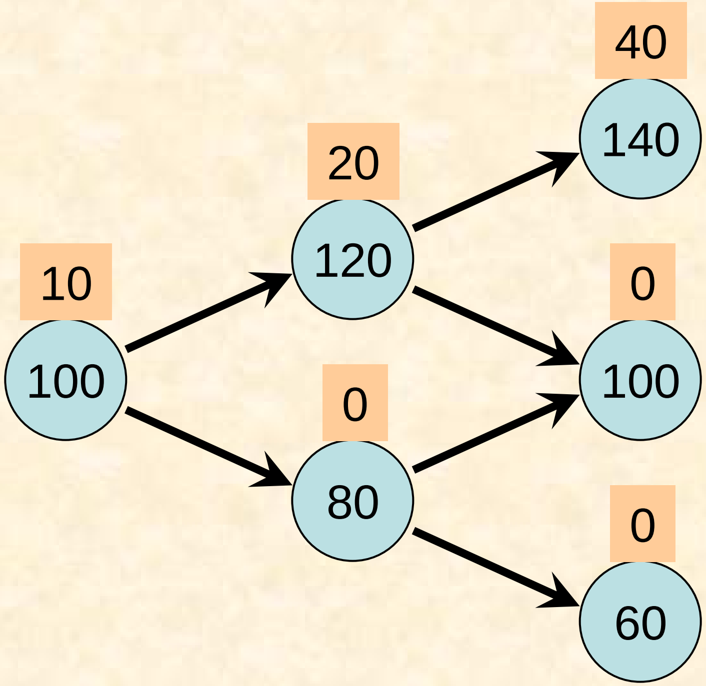


Binomial Tree Model

Call option with maturity at the second tic, with strike of 100$

Again, arbitrage dictates a risk-free strategy and a risk-free price.

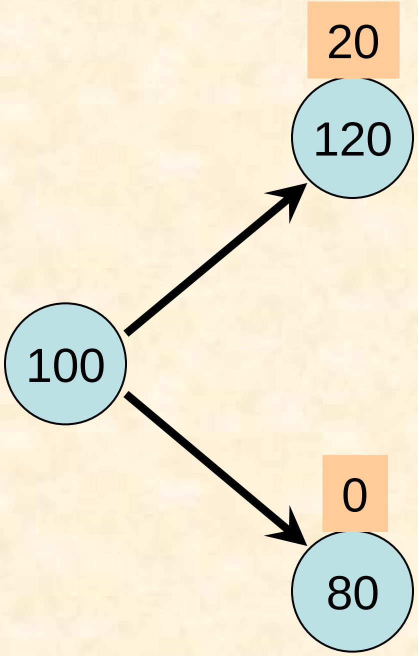


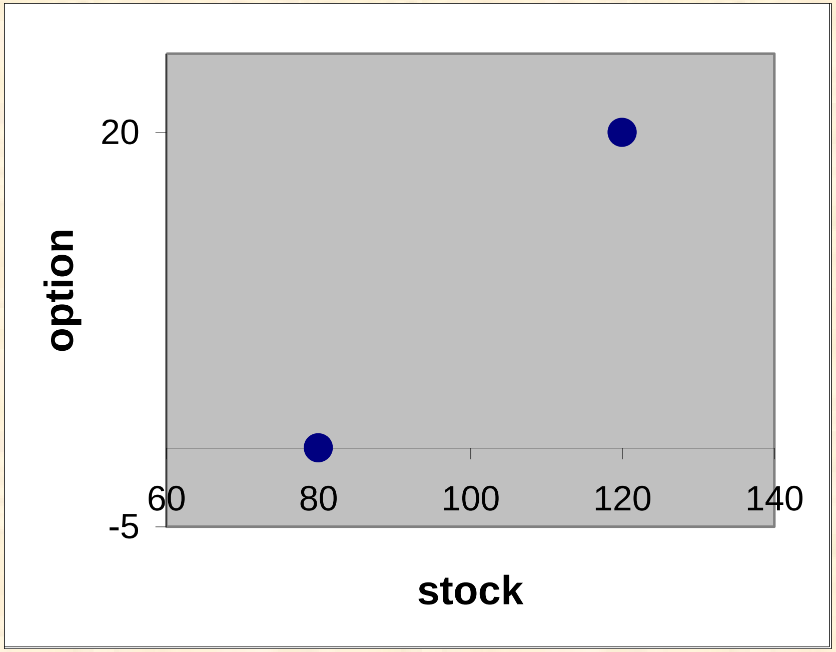
Hedging strategy: Buy and sell stock depending on share price trajectory.

Conclusion of Classical Theory

- Price of option is determined not by perceived probabilities of the stock's price dynamics, but by no arbitrage condition
- There is a hedging strategy that reproduces the option by continuously rebalancing a stock portfolio
- Under quite general conditions, this strategy is risk-free

Does hedging always work ?





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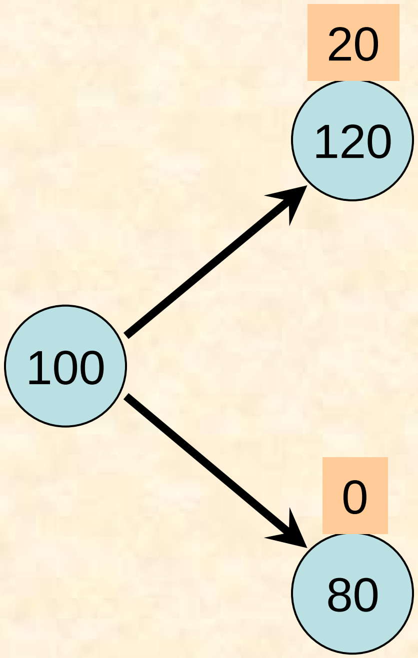

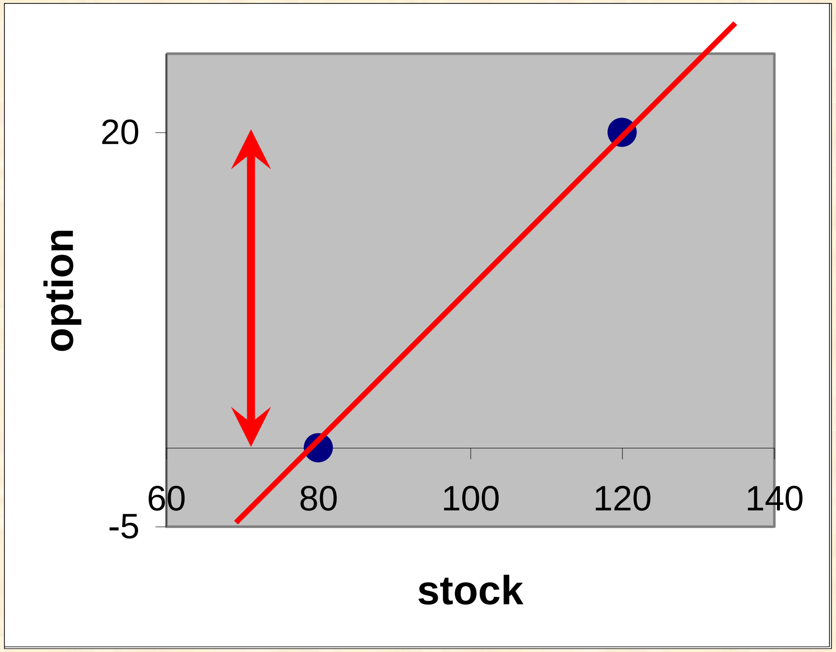

There is a perfect hedge
We can nullify risk

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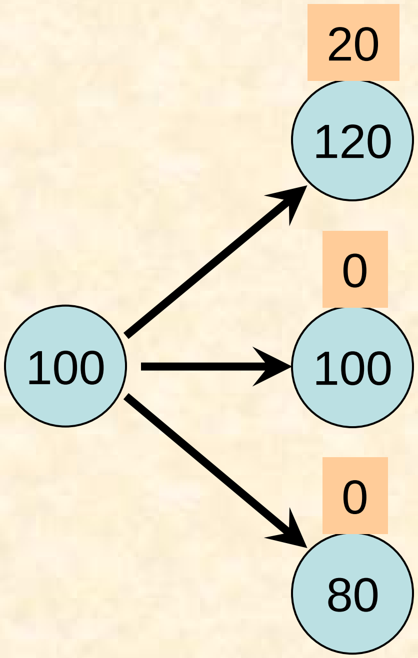

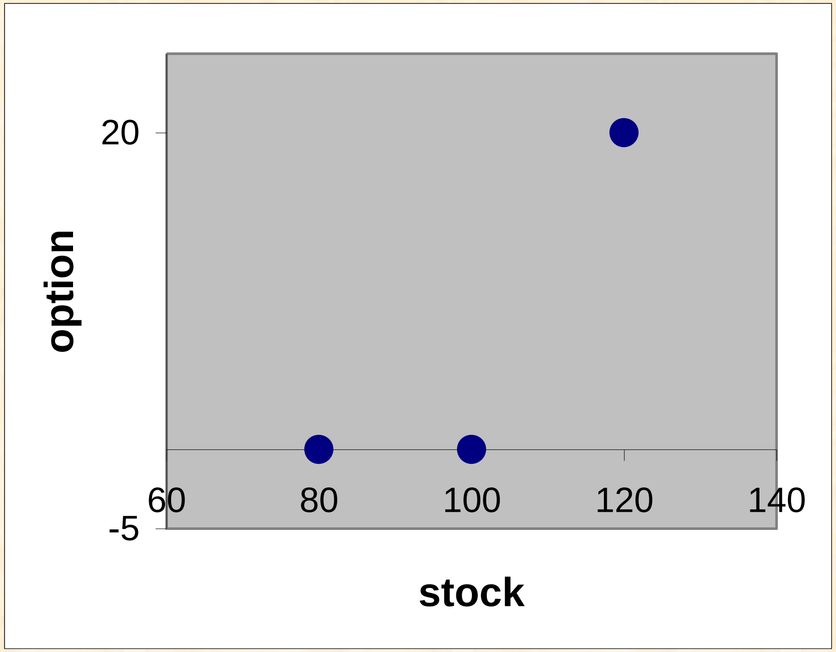

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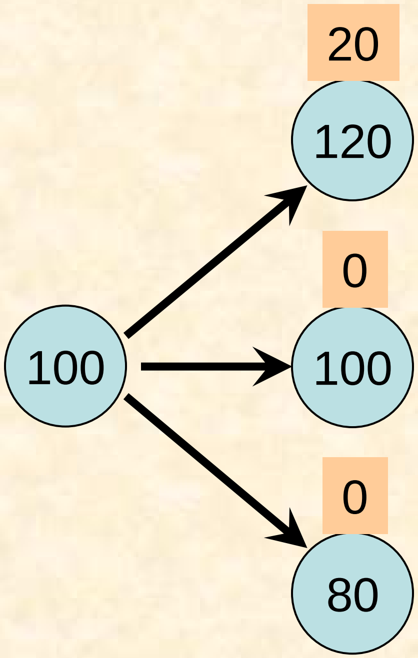
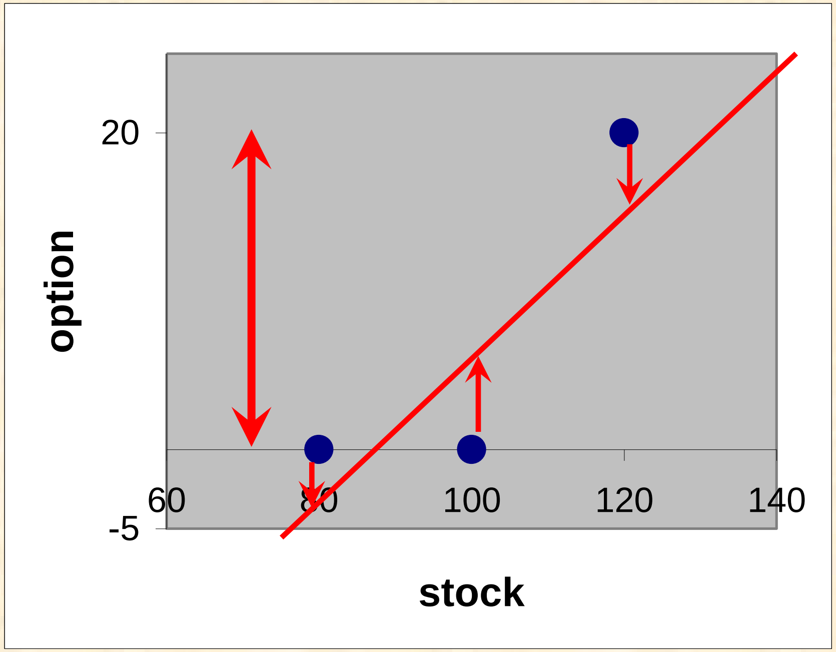
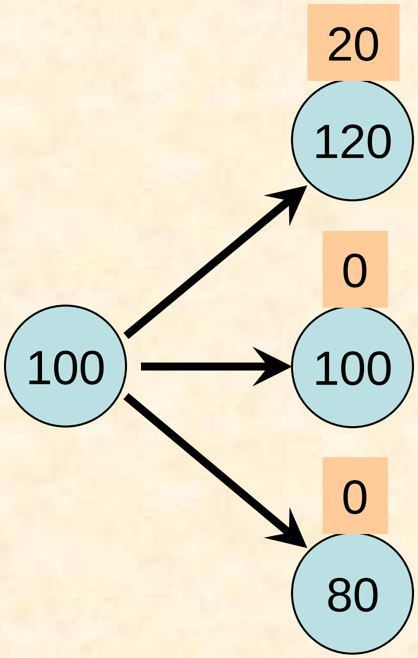
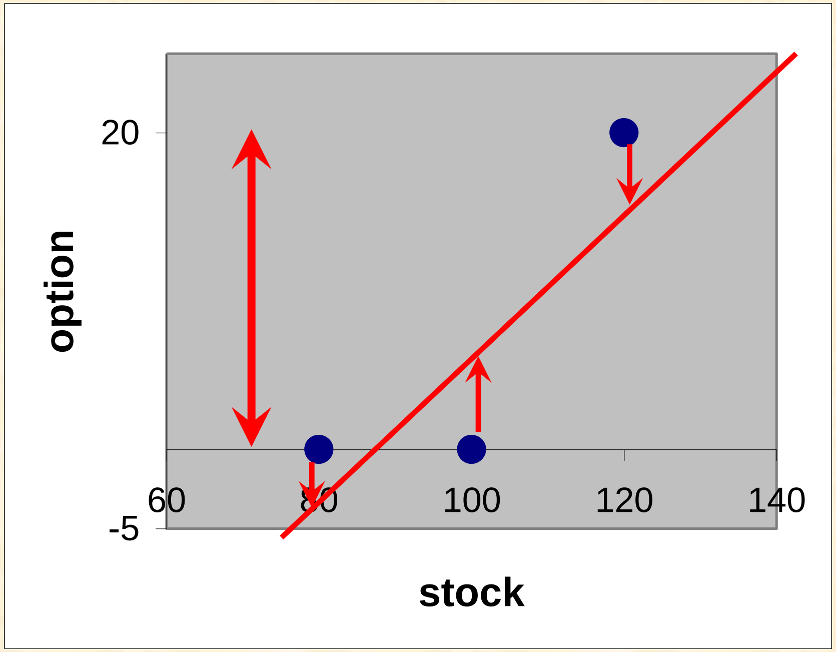




There is no perfect hedge
We can reduce risk, but cannot nullify it

Does hedging always work ?

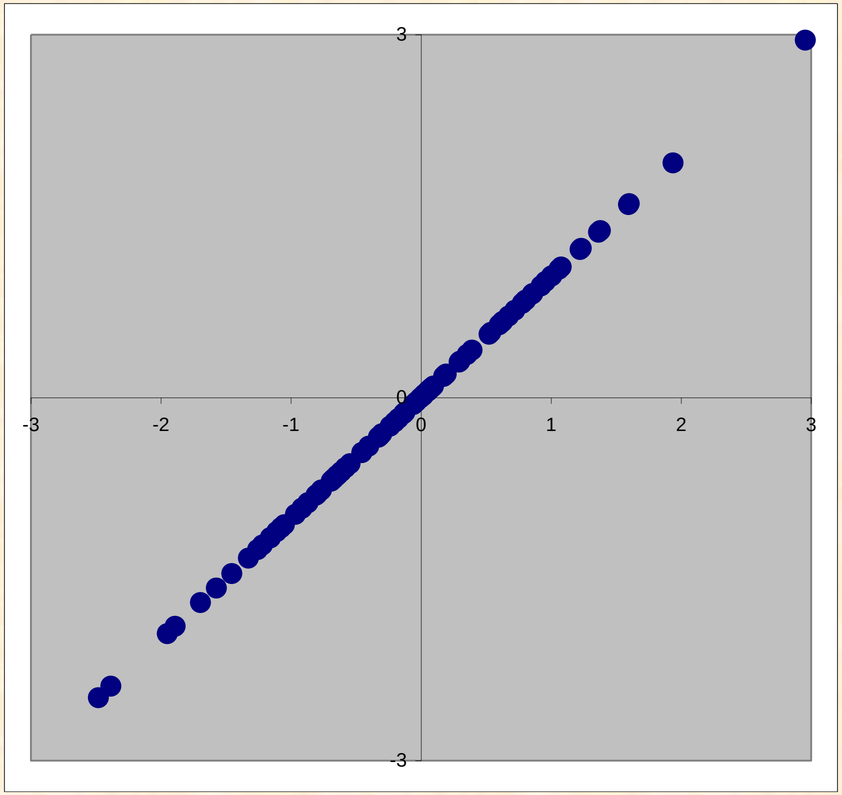




If we define risk as the mean square deviation, then the risk-minimizing strategy is the linear regression line.

General examples

Two random, but perfectly correlated instruments



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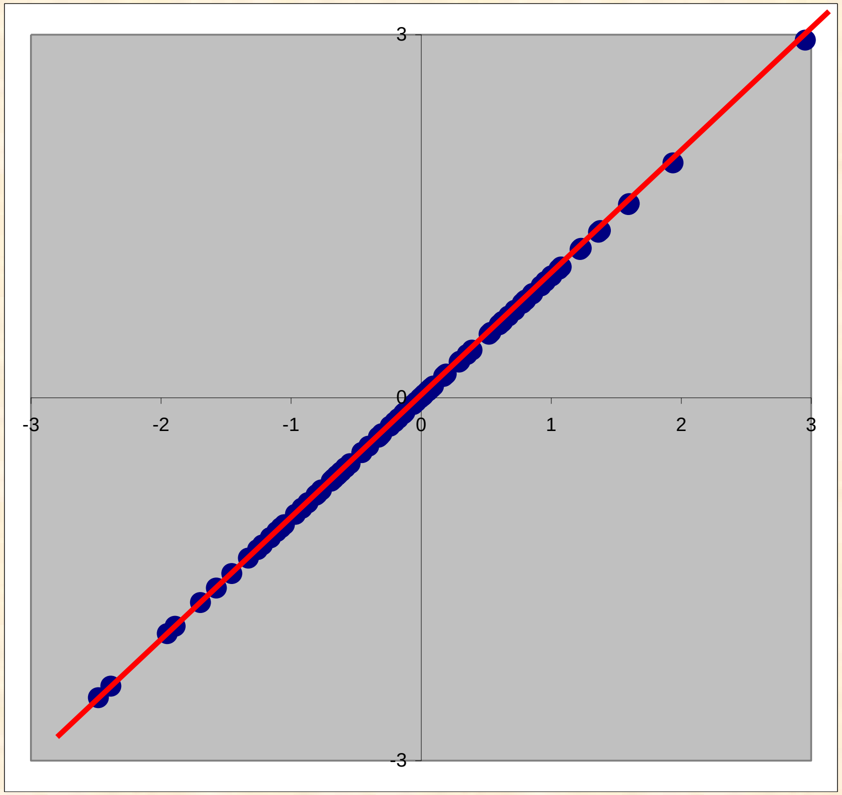

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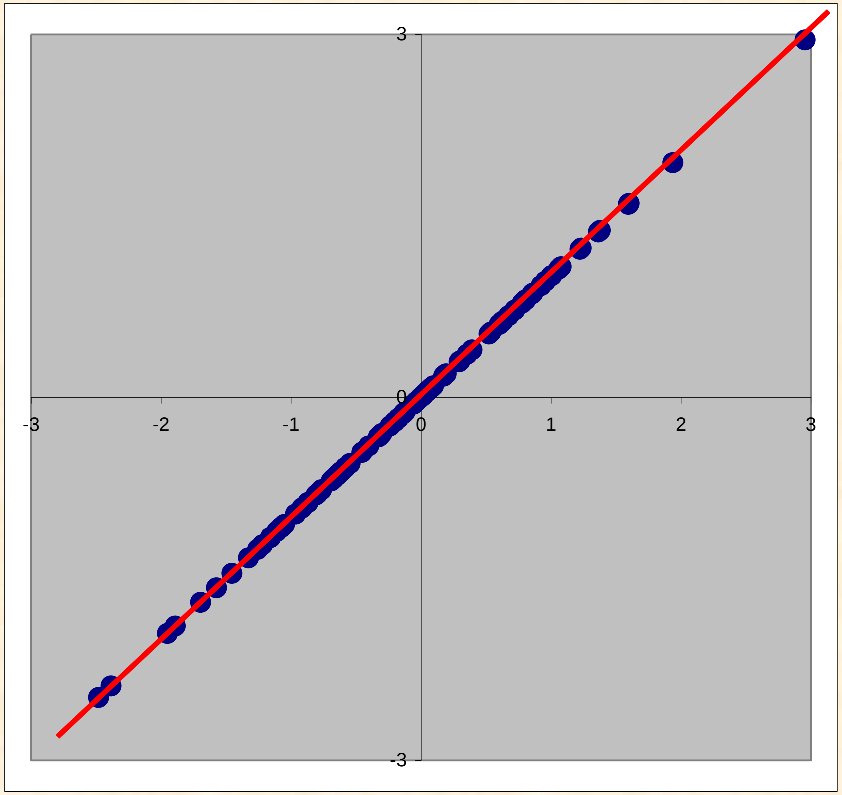

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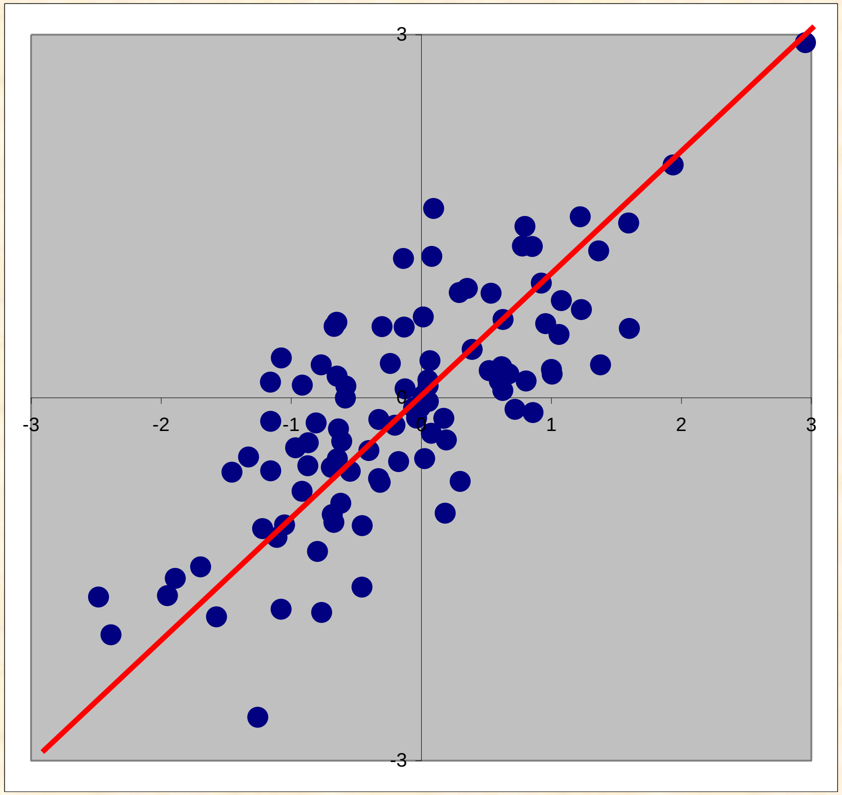
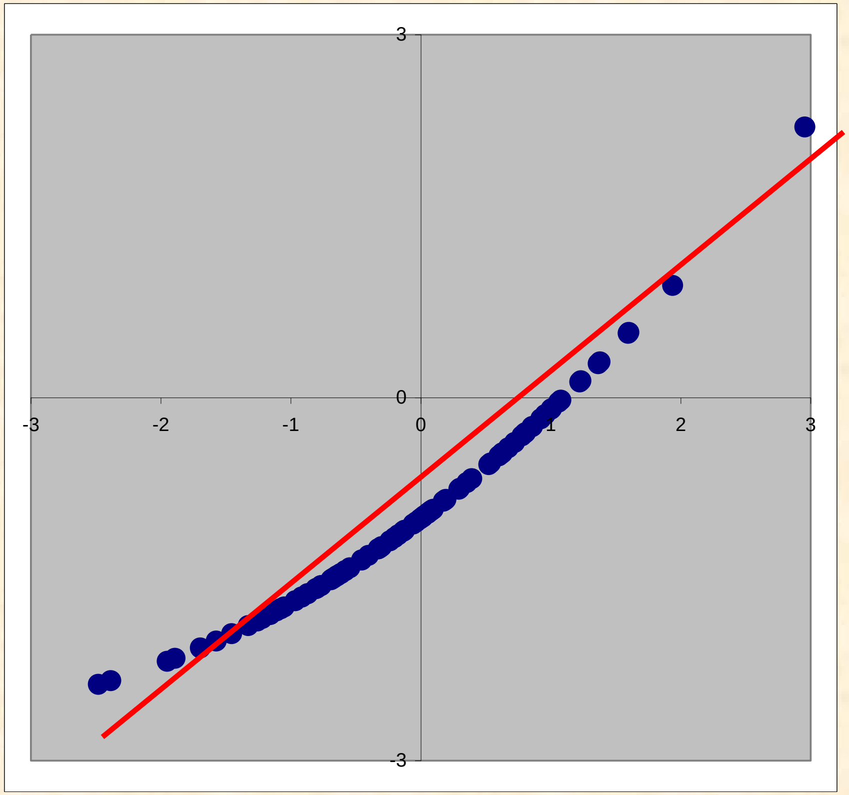


General examples

Two random instruments related
by a non-linear function



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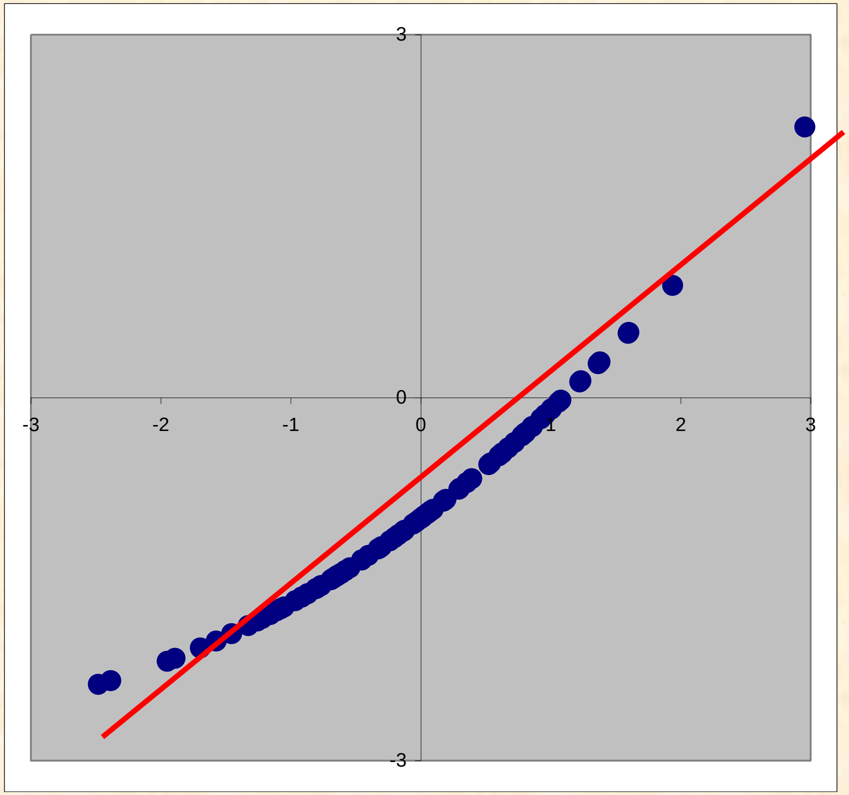
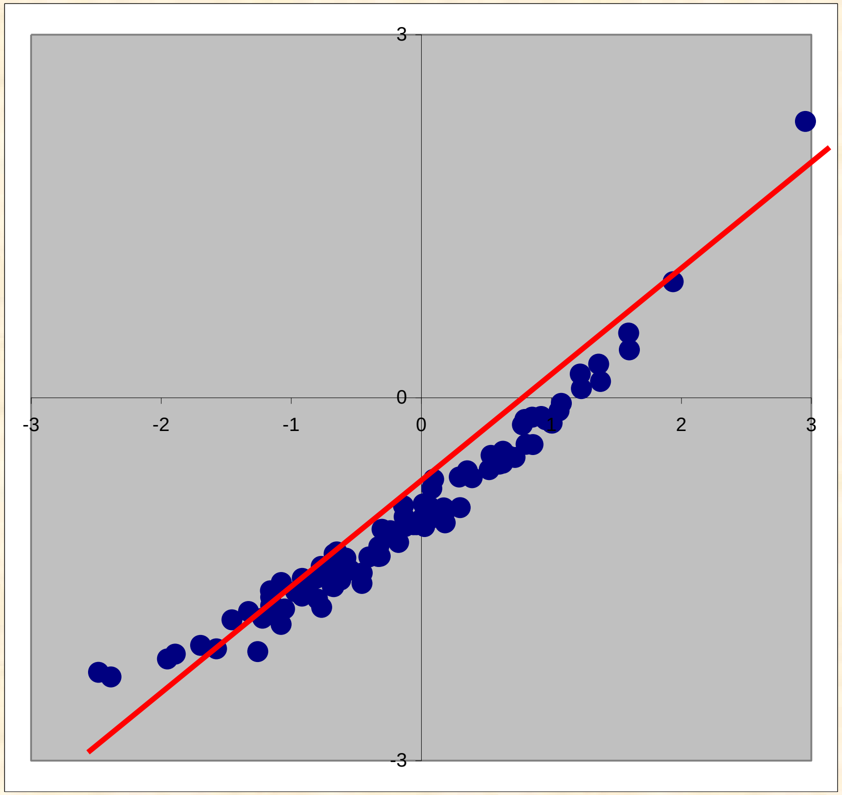
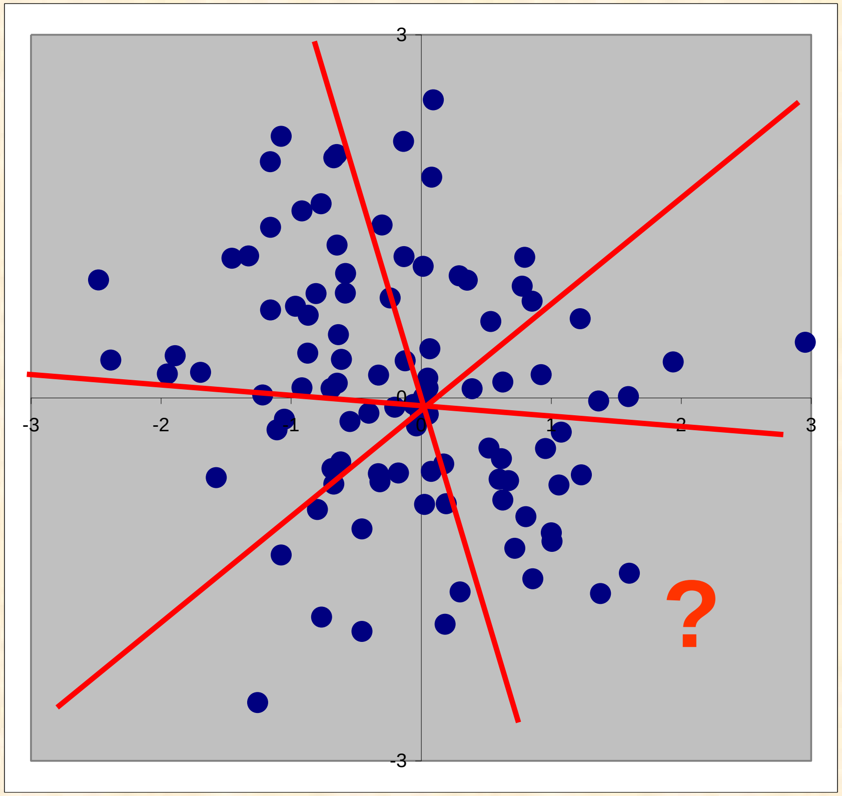


Same, with additional noise



General examples

Two random, perfectly uncorrelated instruments



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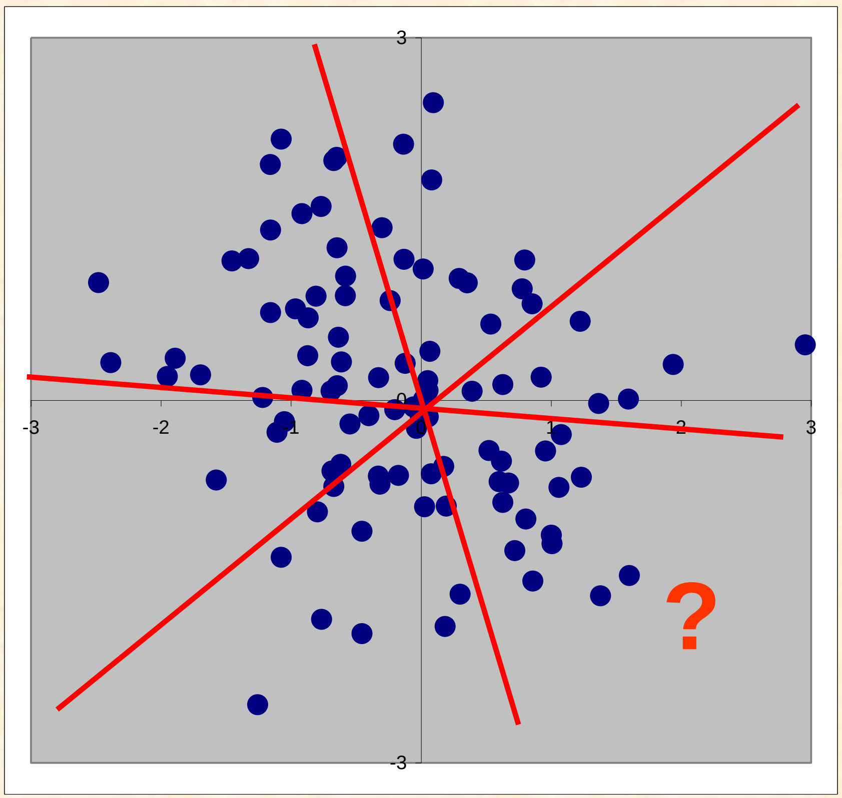

Hedging efficiency
is determined by the
correlation between
the instruments

Conclusions

- Classical theory paints an optimistic picture and restricts itself to complete markets where perfect hedging is possible
- Hedging, if cannot eliminate, at least should minimize risk
- Hedging efficiency depends on the correlation between instruments

References

- Financial Calculus (M. Baxter, A. Rennie)
- Stochastic Calculus For Finance I. (S. E. Shreve)
- Financial Modelling With Jump Processes (R. Cont, P. Tankov)